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$$\int_0^1 x^\alpha \ln\frac{e}{x} f(x)\,dx,\quad \int_0^1 x^\beta \ln\frac{e}{x}\ln\frac{e}{1-x} f(x)\,dx$$
$$\int_0^1 \ln\frac{1}{x} f(x)\,dx,\quad \int_0^\infty x^\beta e^{-x}\ln\left(1+\frac{1}{x}\right) f(x)\,dx$$

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FRACTURE CRITERIA FOR MATERIALS WITH DEFECTS*

L.N. GERMANOVICH and G.P. CHEREPANOV

Classical theories of the strength of materials start from concepts of the existence and uniqueness of fracture surfaces in the space of indpendent loading parameters: on approaching a certain point of this surface from within along any arbitrary loading path, the instant of fracture is fixed by the very same combination of loading parameters. Such are all the strength criteria appied in the strength of materials (in stress space), for instance, Galileo, Poncelet, Coulomb, Tresca, Saint-Venant, Moore, Mises, etc. /1-16/. This concept turned out to be valid even from the viewpoint of fracture mechanics in the case of active loading paths /17/.

Analysis of these concept in the case of two (and more) independent loading parameters and for any loading paths is of interest from the viewpoint of modern fracture mechanics according to which the fracture of real materials is explained by the development of cracks in them from certain initial defects. The most widespread kinds of initial defects here are obviously pores and cracks. Representative of crack and pore materials are concrete, ceramics, composites, mountain rocks and other geomaterials for which the representation of a fracture surface is used extensively at present to describe their strength.

We consider below two problems of fracture mechanics with two independent loading

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parameters: the development of an isolated crack under the effect of concentrated forces and homogeneous tension, homogeneous compression and tension of a porous material. It is shown by these examples that a fracture surface does not generally exist: the set of fracture limit points does not form a surface in the space of loading parameters in the general case, but a certain volume (later called the fracture continuum), i.e., depending on the loading path, and fracture can either occur or not occur at a given point of the continuum.

It is shown that in a numer of cases catastrophe theory can be applied to the analysis of the fracture process. The introduction of a potential function of a specific kind enables us to examine a broad class of multiparametric problems of fracture mechanics within the framework of catastrophe theory. The application of the Drucker postulate and the concept of a fracture surface in investigations of the strength of materials is discussed.

1. An isolated crack subjected to concentrated forces and homogeneous tension. We consider first the best-known example /17/ whose artificiality is redeemed by its simplicity (see Sect.4 also). Let concentrated forces P be applied to the edges of a rectilinear crack of length $2L$ in an infinite body. At infinity the body is loaded by a tensile stress p (Fig.1a). The fracture and the crack are considered to be brittle.

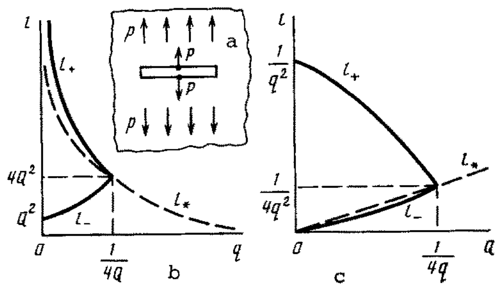

Fig.1

We study the development of the crack in the three-dimensional space L, p, P. The stress intensity coefficient at the end of the crack /18/ equals $K_I = p\sqrt{\pi L} + P/\sqrt{\pi L}$ in this case /17, 18/. For the case of a mobile-equilibrium crack it equals the fracture toughness K_c (a certain constant of the given material /18/).

It is convenient to introduce the dimensionless intensity coefficient $k_I = K_I/K_c$ and the dimensionless crack length $l = L/L_0$, and to take the dimensionless quantities

$$Q = P/(K_c\sqrt{\pi L_0}), \quad q = p\sqrt{\pi L_0}/K_c$$

as independent loading parameters, where L_0 is the initial crack length. Then

$$k_I = q\sqrt{l} + Q/\sqrt{l} \tag{1.1}$$

and later the domain $l > 0$, $q \geqslant 0$, $Q \geqslant 0$ is considered in the space l, q, Q. For a moving-equilibrium crack $q\sqrt{l} + Q/\sqrt{l} = 1$, i.e., its length equals l_+ and l_- where

$$l_\pm(q, Q) = [(1 \pm \sqrt{1 - 4qQ})/(2q)]^2 \tag{1.2}$$

The line of intersection I of the surfaces $l_+(q, Q)$ and $l_-(q, Q)$ lies on a hyperboloid $4qQ = 1$; the crack length l here equals $1/(4q^2)$. We consider a section of the surfaces (1.2) by planes $q = \text{const}$. It is seen that in such a section the curve l_+ is convex while l_- is concave (Fig.1b). In the section $Q = \text{const}$ the curve l_+ decreases from infinity to the value $4Q^2$ while the curve l_- grows from the value Q^2 to the value $4Q^2$ (Fig.1c). The general form of the surface is shown in Fig.2 (the surface $l_+(q, Q)$ is the upper, and $l_-(q, Q)$ the lower).

The condition of stable crack growth, which has the form $\partial K_I/\partial l < 0$ /15/ in the general case, will be the following: $l < Q/q$ in this problem according to (1.1), i.e., the surface $l_*(q, Q) = Q/q$ separating the stability and instability domains is between the surfaces $l_+(q, Q)$ and $l_-(q, Q)$ (the dashed line in Figs.1b and c). All three surfaces have the common line of intersection I; in addition, the surfaces $l_+(q, Q)$ and $l_-(q, Q)$ intersect along the q axis.

Therefore, the surface $l_-(q, Q)$ corresponds to stable crack growth, and the surface $l_+(q, Q)$ to unstable crack development, in this case to the appropriate global fracture (crack propagation to infinity).

The process of crack development in the space l, q, Q can be described as follows. Let

$l = 1$ be the length of the initial crack and let a certain loading process be realized, which is understood to be a change in the parameters q and Q ($q = 0, Q = 0$ at the initial instant), and to which a certain line (Fig.3) corresponds in the plane $l = 1$. When this line is tangent to one of the surfaces $l_+(q, 0)$ or $l_-(q, Q)$, the crack goes over into a moving-equilibrium state. If this is the surface $l_+(q, Q)$ (the dashed line in Fig.3), then fracture is of a catastrophic nature. If this is the surface $l_-(q, Q)$ (the solid line in Fig.3), then under further loading, when

$$(\partial l/\partial q)\, dq + (\partial l/\partial Q)\, dQ > 0 \tag{1.3}$$

stable crack growth is possible and motion of the imaging point occurs along the surface $l_-(q, Q)$. When

$$(\partial l/\partial q)\, dq + (\partial l/\partial Q)\, dQ = 0, \quad dq \leqslant 0, \quad dQ \leqslant 0 \tag{1.4}$$

unloading results in the fact that the imaging point moving in the plane $l = \mathrm{const}$ returns on the l axis (see Fig.2).

It is hence clear that any point within the triangle OAB in Fig.3 can be a point of global fracture since loading can always be realized in such a manner that the imaging point will not return on the surface $l_-(q, Q)$ but will arrive at the surface $l_+(q, Q)$ (the latter corresponds to unstable crack development, i.e., to global fracture).

It should be noted that not taking account of possible unloading paths would of necessity result in the existence of a single fracture surface /17/ in the space q, Q. As is seen, in the general case of any loading paths, a fracture surface does not exist: fracture can occur at any point of the domain F in the q, Q plane for a suitably selected loading path (i.e., the fracture continuum F in this example will be bounded by the lines $q = 0$, $Q = 0$, $qQ = 1/4$, $q + Q = 1$, hatched in Fig.3).

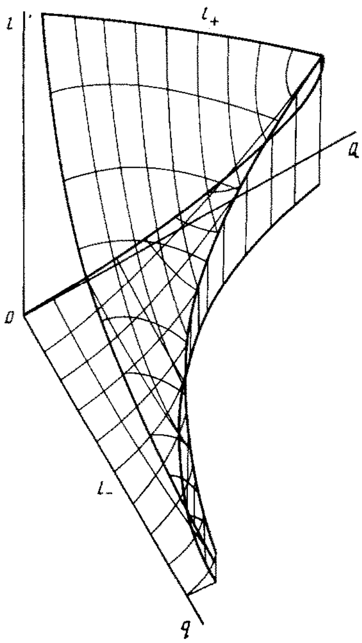

Fig.2

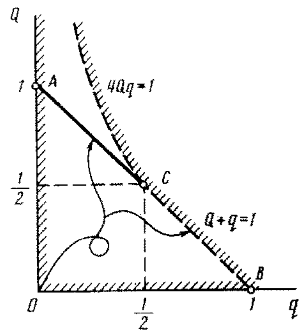

Fig.3

2. Homogeneous compression and tension of a porous material. We consider a problem that can be the basis for the fracture model of porous materials /19/ under biaxial compression-tension.

As is well-known, in the compression of brittle bodies with a hole, stable cracks grow form the latter and are oriented along the line of compressive action. This kind of fracture has been observed both in models /20/ and in real materials /19/.

In an infinite space let there be a cylindrical pore with two crack branches emanating from it along the pore diameter at opposite points of the latter, and let the space be compressed at infinity by a stress $\sigma_1 \geqslant 0$ and stretched by a stress $\sigma_2 \geqslant 0$ oriented as shown in Fig.4a. Let the pore radius be R, and the length of the branches $L - R$. The stress intensity coefficient at the ends of the branches can be written in the following form from dimensional analysis considerations

$$K_I = \sigma_1 \sqrt{2\pi R}\, \lambda_1(l) + \sigma_2 \sqrt{2\pi R}\, \lambda_2(l) \quad (l = L/R) \tag{2.1}$$

Here $\lambda_1(l)$ and $\lambda_2(l)$ are certain functions that are found numerically in a number of papers, for instance in /19, 21/.

We approximate the numerical results of /19, 21/ by the following dependences

$$\lambda_1(l) = \frac{0{,}58\sqrt{l-1}}{(l-0{,}1)^3}, \quad \lambda_2(l) = \sqrt{\frac{l-1}{2}}\, \frac{l-0{,}21}{l-0{,}76}$$

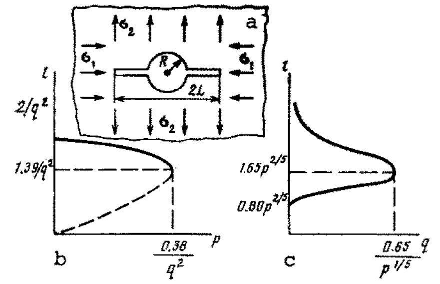

Fig.4

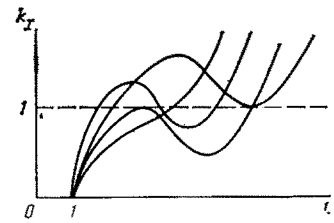

Fig.5

with not more than 6% error for λ_2 and 17% for λ_1 (for the most essential values $1 \leqslant l \leqslant 1.7$ for later, the approximation error for λ_2 does not exceed 10%). We specially do not take too complex and more exact approximations since the purpose is to demonstrate the general properties of phenomenological fracture criteria.

We rewrite (2.1) in the form

$$k_{\rm I} = p\lambda_1(l) + q\lambda_2(l), \quad p \geqslant 0, \quad q \geqslant 0$$
$$(k_{\rm I} = K_{\rm I}/K_c, \quad q = \sigma_2\sqrt{2\pi R}/K_c, \quad p = \sigma_1\sqrt{2\pi R}/K_c)$$

The connection between the independent loading parameters for a moving-equilibrium brittle crack will here have the form

$$p\lambda_1(l) + q\lambda_2(l) = 1 \quad (l > 0,\ p \geqslant 0,\ q \geqslant 0) \tag{2.2}$$

This is a surface S in the three-dimensional space l, p, q. For small crack lengths (i.e., for $l - 1 \ll 1$) the equation of the surface S can be written as follows: $l = 1 + (0.79p + 2.38q)^{-2}$ and this surface is obviously realized for large values of at least one of the parameters p and q. In the case of large crack lengths, sections through the surface S by the planes $q = \text{const}$ and $p = \text{const}$ have the form shown in Fig.4b and c. They are meaningful for large p and small q.

The surface separating the stability and instability domains is given by the equation $p\lambda_1'(l) + q\lambda_2'(l) = 0$ $(\partial k_{\rm I}/\partial l = 0)$ and is easily investigated analytically for large l. The projection of its line of intersection I with S on the l, p plane is given in this case by the equation $l = 1.65p^{2/5}$. The projection I' of the line I on the plane of the parameters p, q for large l has the form $p = 0.65/q^5$, which is meaningful for small q.

In this case, however, unstable crack growth still does not mean catastrophic fracture. In fact, a comparison with the dependence $k_{\rm I}(l)$ for different values of the parameters p and q shows (Fig.5) that those parameters p and q for which (2.2) in l has one or two solutions will correspond to catastrophic fracture. If it has three solutions, then either the moving-equilibrium crack is stable or it grows unstably to a stable length, or (for the greatest value of l) unstable global fracture occurs. The general form of the surface S and the lines I and I' is given in Fig.6. The part K of the surface S bounded by the line I and the lines of intersection of S with the coordinate plane l, p corresponds to stable equilibrium of mobile cracks. The points of the transition of the single-valued domain of the function $l(p, q)$ into the tri-valued domain is a singular point of the surface S, which we denote by M. It is obviously determined from the solution of the system $\partial k_{\rm I}/\partial l = 0$, $\partial^2 k_{\rm I}/\partial l^2 = 0$, which takes the following form in this case

$$p\lambda_1'(l) + q\lambda_2'(l) = 0, \quad p\lambda_1''(l) + q\lambda_2''(l) = 0 \tag{2.3}$$

Equating the determinant of this system to zero, we find l. Then using (2.2) and any of the Eqs.(2.3), we find p and q. We obtain the following coordinates of the point M: $l \approx 1.6$, $p \approx 1.8$, $q \approx 0.84$ as a result of a numerical computation.

The crack development process in this space l, p, q can be described now as follows. Let l_0 be the length of the initial crack and let a certain loading process be realized which we understand to be a change in the parameters p and q by following the exposition in Sect.1 $p = 0, q = 0$ at the initial instant). A certain line T corresponds to the loading process in the plane $l = l_0$. When this line is tangent to the surface S, the crack goes over into the moving-equilibrium state. Motion of the imaging point along the projection T' of the line T in the p, q plane corresponds to the loading process in the space of the loading parameters p, q.

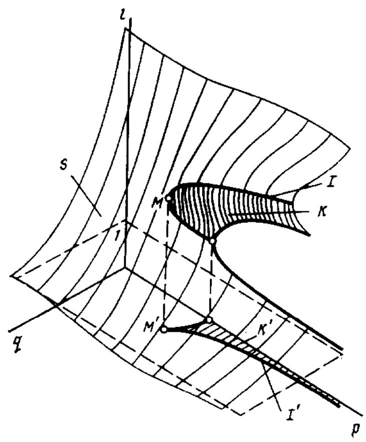

Fig.6

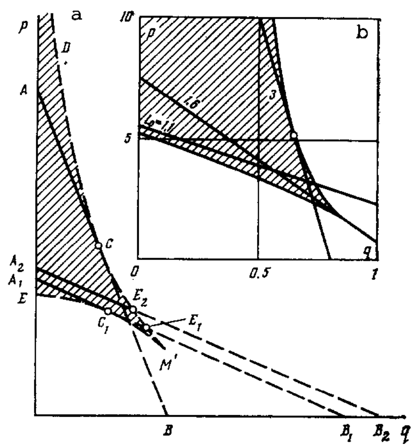

Fig.7

In Fig.7a we show the projection I' of the instability curve I (the line $DM'E$), and the lines $k_I = p\lambda_1(l_0) + q\lambda_2(l_0) = 1$ (the segments AB, A_1B_1, A_2B_2) which when reached by the line T' the crack goes into the moving-equilibrium state.

It is seen that if the length of the initial crack is sufficiently large ($l_0 > a \approx 1.2$) the line $k_I = 1$ is tangent to the line I' at one point. If $l_0 > b \approx 1.6$, the point of tangency C lies on EM' while if $l_0 < b$ the point of tangency C_1 lies on EM'. If $l = b$ the point of tangency is the point M' (as is seen from Fig.7b, where the appropriate construction is made to scale). The line $k_I = 1$ here (A_1B_1 intersects DM' at the point E_1).

If the initial crack length is sufficiently small ($l_0 \leqslant a$) there are no points of tangency and the line $k_I = 1$ (A_2B_2) intersects I' at a point E_2 lying on DM' (excluding the case $l_0 = a$ when E is the second point of intersection).

We will now examine the loading process for a sufficiently long initial crack length: $l_0 > a$. If the imaging point is tangent ot the surface S outside the zone K bounded by the line I and the line of intersection of S with the coordinate l, p plane l, p, q space, then the crack turns out to be in a state of unstable equilibrium and complete catastrophic fracture occurs. If the imaging point turns out to be on the part K of the surface S, equilibrium of the mobile crack will be stable. Under further loading (condition (1.3)), stable crack growth is possible to a certain length $l_1 > l_0$, when the imaging point moves all the time in the domain K (if it reaches the line I, total fracture will occur). Unloading (the relationship (1.4)) enables us to return the imaging point to the l axis in l, p, q space as it moves in the plane $l = l_1$. Under further loading (again condition (1.3)), it is always possible to take the imaging point out to the surface S outside the zone K. Then total fracture occurs.

The following correspond to the process described in the p, q plane. If the imaging point in the loading parameter space is tangent to the line $k_I = 1$ below the point of tangency of this line with the projection K' of the domain K in the p, q plane (i.e., touches the segment BC or B_1C_1), total fracture will occur. In the opposite case (in which the imaging point is tangent to the segment AC or A_1C_1) the crack turns out to be in a stable moving-equilibrium state and will remain there even under further loading when the imaging point moves within the domain K' cross-hatched in Fig.7a. If it touches the upper boundary of the domain K' (DC for $l_0 > b$ and DE_1 for $l_0 \leqslant b$), total fracture will occur. Unloading can be realized differently, by returning the imaging point to the origin.

Now, let $l_0 \leqslant a$. In this case tangency of the imaging point in l, p, q space to the surface S in the single-valued or double-valued domains is equivalent to total fracture (the imaging point in the p, q plane is correspondingly tangent to the segment E_2B_2 of the line $k_I = 1$, i.e., to the line A_2B_2). If it touches S in the tri-valued domain (the segment A_2E_2 in Fig.7a), the crack also starts to be developed unstably but only to a stable length since the imaging point moving parallel to the l axis will again be incident on the surface S but in the stable crack domain K. (The imaging point in the p, q plane does not alter its position since the loading parameters are fixed here.) Now, the crack length is $l > a$ and all its further development is analogous to that described above.

Therefore, any point within the shaded domain in Fig.7 can correspond to one of the following four processes depending on the loading path and the initial crack length: no crack growth; stable crack growth; unstable crack growth to a stable length; and total catastrophic fracture.

It is seen that in the case considered the fundamental deduction of Sect.1 remains valid: in the general case of arbitrary loading paths there is no loading surface but a certain fracture continuum F exists (which is bounded by the coordinate axes, the boundaries of the domain K' and the line $p\lambda_1(l_0) + q\lambda_2(l_0) = 1$ in this example).

3. Application of catastrophe theory in fracture mechanics.

The problems studied show that when there are two or more loading parameters the crack development process is substantially complicated and acquires completely new qualitative features not inherent in the one-parameter problems. Multiparameter problems of a single crack can be considered within the framework of catastrophe theory by using a potential function of the following kind

$$V(c, p_1, p_2, \ldots, p_n) = \int \left[\int_B \Gamma(c, p_1, p_2, \ldots, p_n)\, dB \right] dc - \Gamma_c l \tag{3.1}$$

where c is a variable length dimension (governing the location of the crack front B), $p_1, p_2, \ldots, p_n$ are loading parameters, Γ is the specific energy flux on the crack front which is equal in magnitude to the rate of elastic energy liberation, Γ_c is the critical value of Γ for a moving-equilibrium crack, and dB is the arclength element of the crack front. The quantity Γ is determined by using the stress intensity coefficients $(K_{\rm I}, K_{\rm II}, K_{\rm III})$ on the crack front in the following way /15/:

$$\Gamma = (2\mu)^{-1}[(1-\nu)(K_{\rm I}^2 + K_{\rm II}^2) + K_{\rm III}^2] \tag{3.2}$$

where μ is the shear modulus, and ν is Poisson's ratio of the material in which the crack is located.

Values of the intensity coefficients are found from solutions of the elastic problem; they are certain functions of c and the loading parameters $p_1, p_2, \ldots, p_n$.

According to catastrophe theory /22/, the conditions $\partial V/\partial c = 0$ which result in the equation

$$\Gamma(c, p_1, p_2, \ldots, p_n) = \Gamma_c \tag{3.3}$$

correspond to the equilibrium position.

This is the local fracture condition that is satisfied at the front of a movable equilibrium crack /15/.

Local crack development is stable if $\partial^2 V/\partial c^2 > 0$ (i.e., $\partial\Gamma/\partial c < 0$). If $\partial^2 V/\partial c^2 < 0$ (i.e., $\partial\Gamma/\partial c > 0$), then the local crack development will be unstable. The boundary of the stability and instability domains has the form $\partial^2 V/\partial c^2 = 0$ or equivalently

$$\partial\Gamma/\partial c = 0 \tag{3.4}$$

The set of parameters $p_1, p_2, \ldots, p_n$ governing the unstable crack growth (the catastrophe set /22/) is determined by system (3.3) and (3.4). Since the Hessian of the function V agrees with $\partial^2 V/\partial c^2$ in this case, this means that the system (3.3), (3.4) determines the degenerate critical points of the function V (of non-Morse type), as it should be if it follows catastrophe theory /22/.

Let us examine the examples selected above from this viewpoint by using the terminology of catastrophe theory /22/.

In the second example (Fig.4a, 6, 7) the curve of the folds I of the surface of critical points S of the function V separates the set of Morse and non-Morse critical points of the function V. The projection on the plane of the control parameters p and q is the bifurcation curve I' which always has a singular return point M' (Whitney's theorem on catastrophe mapping), which is a node point: the projection of the beginning of M on the plane of control parameters. Therefore, an elementary canonical node catastrophe is realized in this example. The projection K' of the set K of non-Morse critical points of the function V on the plane of the control parameters (catastrophe set) is bounded by the bifurcation curve I' and is shaded in Fig.7a. Since K' is simultaneously the projection of two "univalent" sets of Morse critical points of the function V, it hence follows that any point of the domain K' on the plane of control parameters can correspond to at least two states of system equilibrium, stable and unstable, depending on the loading path.

In the first of the examples chosen above (Figs.1 and 3), the situation is more simple since the set of non-Morse points is the line of intersection of the surfaces $l_+(q, Q)$ and $l_-(q, Q)$, i.e., this is a fold line (there is no assembly, a catastrophe of fold type).

It should be noted that catastrophe theory cannot completely replace the formulation and solution of boundary value problems in the mechanics of brittle fracture; in a number of cases it is a convenient language of analysis.

4. Limiting fracture surface, the Drucker postulate.

It was shown above that there is generally no limiting fracture surface (FS), but a certain fracture continuum exists. However, the concept of a limiting FS has become widespread in engineering practice (see /1-16/, 23-31/, say, and the bibliography cited here) and attempts continue both to obtain a FS experimentally /25, 27, 28, 31/ and their best description by relying on a formal-geometric construction /9, 12, 28, 31/ and different assumptions of a physical nature /10, 23-27/. The following meaning can obviously be appended to this: if the set of loading paths in the space of the loading parameters is such that the fracture continuum is a sufficiently thin layer or degenerates into a surface, then it is possible to speak about the concept of a limiting FS by keeping in mind, however, just this set of loading paths and not any other. Thus, in certain active loading cases, the fracture continuum actually degenerates into a surface (see /17/). However, in the general case of the fracture of materials with defects it is impossible to consider such a deduction to be sufficiently well founded. Indeed, since the geometry of defect (crack) development depends on the loading path, there are no sufficient foundations to expect degeneration of the fracture continuum into a FS in the general case, even for active loading.

Note that the method of constructing strength criteria for a given kind of loading has already been encountered in engineering practice /24/.

Convexity of the limiting FS /7, 10, 26/ is usually postulated when constructing different theories of strength (the Drucker postulate /32/, non-compliance with it is often considered a basis for acknowledging the strength criterion to be erroneous) although there are individual doubts about the validity of this situation /9, 31/. It is clear from the above that the FS can even be concave with any radius of curvature. Indeed, any geometric surface (convex, concave, or planar) located in the fracture continuum, can be a FS for any suitably selected set of loading paths.

In this connection, it should be emphasized that in the specific examples considered above, a two-dimensional fracture continuum was obtained and not a one-dimensional FS because not only loading (1.3) but also unloading (1.4) was achieved. However, in the case of loading the FS (now completely defined) is concave. This is seen clearly in both the first exmaple (see Fig.3 presented above and Fig.8 in /17/) and the second example (Fig.7b).

Therefore, transfer of the Drucker postulate from plasticity theory to the theory of strength does not have sufficient foundations and can result in error in a number of cases.

In conclusion we consider still another generalization of the approach developed above to the questions selected. In addition to the classical space Θ of the loading parameters, we introduce a space of all parameters of the problem Π. Thus in the first of the examples considered $\Theta = \{q, Q\}$, $\Pi = \{l, q, Q\}$, while in the second example $\Theta = \{p, q\}$, $\Pi = \{l, p, q\}$. Therefore, Π includes not only the loading parameters but also parameters of the material defects, where the space Θ of the loading parameters is a subspace of Π.

We introduce the FS S_Π for materials with growing defects in the space Π. This surface divides the space Π into two parts and is a set of points for which the condition of defect development (condition (3.3), say) is satisfied. As follows from the examples considered, its dimensionality is $\dim S_\Pi = \dim \Pi - 1$.

For such a definition the FS is completele definite and unique. It is divided into three domains: the domain of total fracture of the body as a whole, the domain of unstable growth of defects with a subsequent arrest, and the domain of stable defect development, where the last two domains can even be missing.

In the first example $S_\Pi = l_1 \bigcup l_2$, and in the second $S_\Pi = S$.

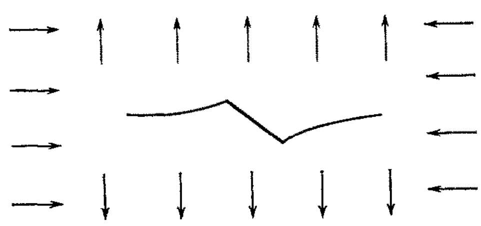

Fig.8

According to classical representations, the FS is inserted into the space Θ and is a set of points S_Θ corresponding to total fracture of the body, and its dimensionality is $\dim S_\Theta = \dim \Theta - 1$. This paper shows that this is not always true in general; cases are possible for which $\dim S_\Theta = \dim \Theta$. In this case S_Θ is called the fracture continuum F above.

In a number of cases, catastrophe theory can be applied to analyse the projection of S_Π on Θ (by using the potential function introduced above, say), which will often (see /22/) facilitate analysis of multiparametric problems encountered in applications since the whole diversity of real situations is reduced successfully to a moderate number of well-studied schemes by using catastrophe theory. In the case of two-parameter problems only catastrophes of the node and fold type /22/ are possible. Consequently, the particular problems considered above have a common nature and reflect the fundamental qualitative features of the two-parameter loadings that result in fracture.

Thus, the generalized fracture surface for tension-compression of cracking material (Fig. 8) is qualitatively similar to that displayed in Fig.2. This follows from the fact that an inclined crack in a homogeneous material grows stably under compression, being oriented along the compression axis /33-35/ (in certain cases /36/* (*See also: Dyskin A.V., Effective deformation characteristics and fracture conditions for solid bodies with a system of oriented cracks: Candidate Dissertation, Inst. Problems of Mechanics, USSR Acad. Sci., 1986 (where the idea from /36/ is developed and used successively) a crack growing stably under compression is modelled by a slit whose edges are loaded by a pair of concentrated forces as in the first example examined above), and unstably under tension oriented perpendicular to the tension axis /37/. Consequently, in this case we have a fold-type catastrophe. If, however, the crack under compression grows unstably at the initial instant, then a node appears on the fold (Fig.6).

Let us note again that the dimensionality of the fracture continuum is $\dim F = \dim \Theta = 2$ in the examples considered, which is due to the possibility of stable crack growth. In these cases when stable crack growth is impossible /38/, fracture will be determined only by the global fracture surface into which F has degenerated and which can consequently even be convex (non-concave) /38/. Namely, the possibility of stable crack growth results in a change in their lengths, i.e., a change in the parameter of the material structure which is therefore a latent parameter in the classical approach.

Transfer of the Drucker postulate to the case of assigning the fracture surface S_Π in the space Π is not, strictly speaking, legitimate as is seen clearly from an examination of Figs.2 and 6. However, this is a common deduction that follows, in particular, from the geometry of assembly-type catastrophes /22/. Moreover, it follows from the examples selected that more often the concavity postulate, which can be formulated as follows: a non-degenerate fracture continuum $F\,(\dim F = \dim \Theta)$ is bounded by a concave (non-cave) surface S_F, is valid for a fracture continuum rather than the convexity postulate.

Other physical reasons for the concavity of S_Π and S_F, different from those considered here, will be the subject of further study.

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